PURPOSE

The purpose of this exercise is to study the limitations of conventional beam bending relations applied to curved beams and to use photo elasticity to determine the actual stresses in a curved beam for comparison to analytical and numerical solutions.

The exercise has two main efforts: 1) Experimental Procedures to determine the fringe values for the straight beam of "calibrating" the birefringent test material and 2) Work Sheet calculations of stresses for comparison of analytically-determined stresses with experimental (photoelastic) and numerical (FEA) results.

EQUIPMENT

• Straight beam of a birefringent material.
• Curved beam of the same birefringent material as the straight beam.
• Four-point flexure loading fixture with load pan and suitable masses (straight beam)
• Line-loading fixture with load pan and suitable masses (curved beam)
• Circular polariscope with monochromatic light source

EXPERIMENTAL PROCEDURES

Procedure 1. Straight Beam in Pure Bending to Determine ("calibrate") the Stress-Optical Coefficient of the Material

i) Install the straight beam (see Fig. 1) in the four-point flexure loading fixture
ii) Attach the load pan (Note: The combined pan/fixture mass is ~0.980 kg)
iii) Apply two 10-kg masses one at a time to the load pan.
iv) With polarizer and analyzer crossed (dark field), focus the camera, record the image.
v) Determine the maximum fringe orders at the top and bottom of the beam including estimates of fractional fringes orders by counting the fringes.
vi) The stress-optical coefficient can be calculated using the following relation:

\[ f = \frac{t}{N} (\sigma_1 - \sigma_2) \]  

where \( f \) is the stress-optical coefficient, \( N \) is the average fringe order, \( t \) is the model thickness, and \( \sigma_1 \) and \( \sigma_2 \) are the plane-stress principal stresses.

Procedure 2. Curved Beam in Tension and Bending

i) Install the curved beam (see Fig. 2) in the line loading fixture
ii) Attach the load pan (Note: The combined pan/fixture mass is ~0.454 kg)
iii) Apply one 5-kg mass to the load pan. (Note: Do not apply more than 5 kg at any time).
iv) With polarizer and analyzer crossed (dark field), focus the camera, record the image.
v) Determine the maximum fringe orders at point A at the inside of the straight part of the "arms," at point B at the inside of the curve, and at point C at the outside of the curve.
vi) The stress in the beam can be calculated using the relation:

\[ (\sigma_1 - \sigma_2) = f \frac{N}{t} \]  

where \( f \) is the stress-optical coefficient determined previously, \( N \) is the fringe order, \( t \) is the model thickness, and \( \sigma_1 \) and \( \sigma_2 \) are the plane-stress principal stresses.

* REFERENCES

BACKGROUND FOR RESULTS

When loads are applied to a solid body, such as part of a structure or a machine component, stresses which vary from point to point, are set up in the body. By combining an understanding of engineering statics and mechanics of materials for planar elements (that is, beams) subjected to lateral loading (that is, bending) the well known beam bending relations can be developed (assuming pure bending, constant cross section, linear elastic material, and initially straight beam):

\[
\varepsilon_x = -\frac{y}{\rho} \\
\sigma_x = -E\frac{y}{\rho} = -Ey \frac{M}{E\int y^2 dA} = -\frac{My}{I}
\]

where \( \varepsilon_x \) is the normal strain in the x-direction (longitudinal as shown in Fig. 1), \( y \) is the vertical direction and distance from the neutral axis (transverse as shown in Fig. 1), \( \rho \) is the radius of curvature of the neutral axis due to bending, \( \sigma_x \) is the normal stress in the x-direction, \( E \) is the elastic modulus, \( M \) is the applied moment, and \( I=\int y^2 dA \) is the moment of inertia with respect to the z-axis.

The relations developed in Eq. 1 assume among other things that all longitudinal elements have the same initial length (for example, a “straight beam”). These assumptions lead to the linear variation of strain across the cross section (that is, \( \varepsilon_x = -y/\rho \)). A more general case of beam bending relations can be developed for the case of initially bent beam (assuming pure bending, constant cross section, linear elastic material and a constant initial radius of curvature):

\[
\varepsilon_x = -\left(\frac{r_0 \varepsilon_o}{r}\right) \frac{y}{h_t} \\
\sigma_x = -\left(\frac{r_0 \sigma_o}{r}\right) \frac{y}{h_t} = -\frac{My}{(y+R)Ae} = -\frac{My}{Ae(R-y)}
\]

where \( \varepsilon_x \) is the normal strain in the x-direction (longitudinal as shown in Fig. 2), \( r_0 \) is the outer radius of the initially curved beam, \( r \) is the variable for the radius of the point in question, \( h_t \) is the height of the tensile section of the beam, \( \varepsilon_o \) is the longitudinal strain at the outer surface of the initially curved beam (that is, \( r=r_0 \)), \( y \) is the vertical direction and distance from the neutral axis (transverse as shown in Fig. 2), \( \sigma_x \) is the normal stress in

![Figure 1 Nomenclature for a straight beam with rectangular cross section in pure bending.](image-url)
the $x$-direction, $\sigma_o$ is the longitudinal stress at the outer surface of the initially curved beam (that is, $r = r_o$), $M$ is the applied moment, $R$ is the radius of curvature of the neutral axis, $Ae$ is the first moment of the first section (in this case the tensile section) about the neutral axis such that $Ae = \int ydA$. (Note that $e$ can also be thought of as the distance from the centroid of the first section to the neutral axis of the cross section [a.k.a. eccentricity such that $e = r_i - R$ where $e = \text{centroid} = (r_o + r_i)/2$]). For a rectangular cross section, $R = \frac{r_o - r_i}{\ln(r_o/r_i)}$ but in general can be found by solving the relation $\int \frac{r - R}{r} dA = 0$.

The mathematical solutions for strains and stress in beams (Eqs 1 and 2) provide valuable information regarding the stress distributions in beam-like components with simple geometries and loadings. In more complicated problems, commercially available two- and three-dimensional computer programs for finite element and boundary element analyses (FEA and BEM, respectively) can be used to determine and visualize stress distributions.

These theoretical and numerical results are exact solutions to problems which may or may not model the actual situations (usually due to assumptions about loads, load applications and boundary conditions). This uncertainty in modeling often requires experimental verification by spot checking the analytical or numerical results. A frequently cited example involves a threaded joint which seldom produces uniform contact at the threads. Contact analyses based on the idealized boundary condition of uniform contact will grossly underestimate the actual maximum stress concentration at the root of the overloaded thread. The uncertainty in the contact condition requires a stress analysis of the actual threaded joint experimentally despite the proliferation of FEA and BEM programs. Experimental stress analysis is also necessary to study nonlinear structure problems involving dynamic loading and/or plastic/viscoplastic deformations. Available FEA programs cannot provide detailed stress analysis of three-dimensional dynamic structures. Constitutive relations for plastic/viscoplastic materials are still being developed.

One such experimental procedure often applied to empirically determine stress states is photoelasticity. Photoelasticity is a relatively simple, whole-field method of elastic stress analysis which is well suited for visually identifying locations of stress concentrations. In comparison with other methods of experimental stress analysis, such as a strain gage technique which is a point measurement method, photoelasticity is inexpensive to operate and provides results with minimum effort.

Photoelasticity consists of examining a model similar to the structure of interest using polarized light. The model is fabricated from transparent polymers possessing special optical properties. When the model is viewed under the type (but not necessarily magnitude) of loading similar to the structure of interest, the model exhibits patterns of fringes from which the magnitudes and directions of stresses at all points in the model can be calculated. The principle of similitude can be used to deduce the stresses which exist in the actual structure.
A disadvantage of photoelasticity is the necessity to test a polymer model which may not be able to withstand extreme loading conditions such as high temperature and/or high strain rates. Although photoelasticity is generally applied to elastic analysis, limited studies on photo plasticity and photo viscoelasticity indicate the potential of extending the technique to nonlinear structural analysis. Further details of photoelasticity can be found in listed references.

In this exercise, show all work and answers on the Worksheet, turning this in as the In-class Lab report.
1) Confirmation of Birefringent Test Material: The properties of two birefringent polymers often used for photoelasticity are shown in Table 1. Note which material is used for these laboratory exercises.

<table>
<thead>
<tr>
<th>Homolite 911 a.k.a., CR-39 (allyl diglycol)</th>
<th>Epoxy (Araldite, Epon)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Selected Properties (R.T.)</strong></td>
<td><strong>Selected Properties (R.T.)</strong></td>
</tr>
<tr>
<td>Elastic Modulus, $E$ (GPa)</td>
<td>1.7</td>
</tr>
<tr>
<td>Proportional Limit $\sigma_o$ (MPa)</td>
<td>21</td>
</tr>
<tr>
<td>Poisson's ratio, $\nu$</td>
<td>0.40</td>
</tr>
<tr>
<td>Stress Optical Coefficient, $f$ (kN/m)*</td>
<td>16</td>
</tr>
<tr>
<td>Figure of Merit $Q = \frac{E}{f} (1/m)$</td>
<td>106,250</td>
</tr>
</tbody>
</table>

* in green light with wavelength 546 nm

2) Confirmation of Dimensions: For the two beams and loading fixtures, confirm the following information. See Figs. 3, 4, and 5 for nomenclature and values.

<table>
<thead>
<tr>
<th>Table 2 Dimensions and Loading for Straight and Curved Photoelastic Beams</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Straight beam</strong></td>
</tr>
<tr>
<td>Calibration Force, $F_o$ = $(M_{\text{weight}} + M_{\text{fixture}} + M_{\text{pan}}) \cdot g$ (N)</td>
</tr>
<tr>
<td>Outer Span, $L_o$ (mm)</td>
</tr>
<tr>
<td>Inner Span, $L_i$ (mm)</td>
</tr>
<tr>
<td>Height (straight), $h_s$ (mm)</td>
</tr>
<tr>
<td>Thickness, $b$ (mm)</td>
</tr>
<tr>
<td>Average Radius, $r = \frac{r_o + r_i}{2}$ (mm)</td>
</tr>
<tr>
<td>Height (curve), $h_c$ (mm)</td>
</tr>
</tbody>
</table>

Note: The calibration and test forces must include the masses of the fixture and pan as well as the added masses (a.k.a. weights) in kg. Gravitational constant is $g = 9.816 \text{ kg} \cdot \text{m/s}^2$. 

Figure 3 Nomenclature for the Beams
Figure 4  Straight Beam

Figure 5  Curved Beam
3) **Determination of Stress Optical Coefficient for Material and Setup**

A unique aspect of the four-point flexure loading arrangement is that the region of interest (the section of the beam within the inner loading span) undergoes a pure bending moment as shown in Fig. 6.

![Figure 6](image)

**Figure 6** Free Body, Shear and Moment Diagrams for Four-Point Flexure Loading

**a)** For the straight beam, determine the following if at the outer free edge of the beam \((y=c=h/2)\) the stress state is uniaxial.

Moment of Inertia for the rectangular cross section beam, \(I = \frac{bh^3}{12} = \) \(\text{mm}^4\)

Maximum moment due to calibration force, \(F_o, M_o = \frac{F_o(L_o - L_i)}{4} = \) \(\text{N}\cdot\text{mm}\)

Maximum distance to outer edge of the beam from neutral axis, \(c=h/2 = \) \(\text{mm}\)

Maximum uniaxial bending stress at the outer free edge of the beam

\[\sigma_1 = \sigma_x = \frac{M_o c}{I} = \] \(\text{MPa}\).

**b)** The photoelastic relation can be used to determine the stress optical coefficient directly from the beam bending relation.

From the Experimental Procedure, the average fringe value at the upper and lower outer edges of the beam determined at the calibration force is \(N=\) \(\) \(\text{fringe}\).

Calculated stress optical coefficient for the material, \(f = \frac{b}{N} (\sigma_1) = \) \(\text{MPa-mm/fringe}\).

4) Compare the calculated value of the "calibrated" stress value to that shown in **Table 1** for the material used in this exercise. How do the values compare? Discuss any discrepancies and possible reasons (Note: Do not panic if the calculated stress optical coefficient differs from the value listed in **Table 1**...differences in optical test setup, environmental effects in the material, etc. all require the "calibration" of the material).
5) Experimentally-Measured Stresses in the Curved Beam Using Photoelasticity

At the free edges of selected locations of the curved beam (A, B and C in Figure 3b), the stress states are uniaxial and the photoelastic relation can be used to calculate the normal stresses using the relation between the fringe order at the free edge, the stress optical coefficient for the material, and the specimen thickness.

Fill in the table with values for the loaded test specimen that are used in the following calculations.

<table>
<thead>
<tr>
<th>Fringe Value Counted at A, ( N_A )</th>
<th>Fringe Value Counted at B, ( N_B )</th>
<th>Fringe Value Counted at C, ( N_C )</th>
<th>Thickness, ( b ) (mm)</th>
<th>Calculated Stress Optical Coefficient, ( f ) (MPa-mm/ fringe)</th>
</tr>
</thead>
</table>

a) At "A", the fringe value at the free edge of the curved beam determined at the test force is \( N_A = \) _______.

The photoelastically-determined total normal stress at "A" is \( \sigma_A = \frac{fN_A}{b} = \) _______ MPa.
(Note, apply the appropriate sign(‘+’=tension,’ –’ =compression based on observation)

b) At "B", the fringe value at the free edge of the curved beam determined at the test force is \( N_B = \) _______.

The photoelastically-determined total normal stress at "B" is \( \sigma_B = \frac{fN_B}{b} = \) _______ MPa.
(Note, apply the appropriate sign(‘+’=tension,’ –’ =compression based on observation)

c) At "C", the fringe value at the free edge of the curved beam determined at the test force is \( N_C = \) _______.

The photoelastically-determined total normal stress is \( \sigma_C = \frac{fN_C}{b} = \) _______ MPa.
(Note, apply the appropriate sign(‘+’=tension,’ –’ =compression based on observation)
6) Analytically-Determined Stresses Using Straight Beam Relations

One approach to analytically determine stresses in the curved beam is to assume that the relations for straight beams apply (that is, Eq. 1).

Fill in the table with values for the loaded test specimen that are used in the following calculations.

<table>
<thead>
<tr>
<th>Applied Test Force, $F_t$ (N)</th>
<th>( \text{Length of Straight Leg, } \ell \ (\text{mm}) )</th>
<th>( \text{Height of Straight Leg, } h_s \ (\text{mm}) )</th>
<th>( \text{Thickness of Straight Leg, } h \ (\text{mm}) )</th>
<th>( \text{Outer Radius, } r_o \ (\text{mm}) )</th>
<th>( \text{Inner Radius, } r_i \ (\text{mm}) )</th>
<th>&quot;Height&quot; of Curve, ( h_c ) (mm)</th>
<th>( \text{Thickness of Curve, } b \ (\text{mm}) )</th>
</tr>
</thead>
</table>

a) At "A", the moment is determined as the applied force, $F_t=_____\text{N}$ multiplied by the length of the leg, $\ell=_____\text{mm}$ such that $M_A= F_t \ell =_____\text{N} \cdot \text{mm}$.

The moment of inertia is calculated from the height of the beam, $h_s$, and the thickness of the beam, $b$, such that $I_A = \frac{bh^3}{12} = _____\text{mm}^4$.

The distance from the neutral axis to point "A" (+y is up) is $c = -h_s/2 = _____\text{mm}$.

The normal stress at "A" for a straight beam assumption is $\sigma_{A,\text{straight}} = \frac{-M_A c}{I_A} = _____\text{MPa}$.

(Confirm that the normal stress at "A" should be tension (i.e. $+\sigma$))

b) At "B", the moment is determined as the applied force, $F_t$, multiplied by the length of the leg, $\ell$, mm plus the average radius, $\bar{r} = (r_o + r_i)/2 = _____\text{mm}$ such that $M_B= F_t(\ell + \bar{r}) = _____\text{N} \cdot \text{mm}$.

The moment of inertia at B-C is calculated from the height of the beam, $h_c$, and the thickness of the beam, $b$, such that $I_{BC} = \frac{bh^3}{12} = _____\text{mm}^4$.

The distance from the assumed neutral axis (centroid) to point "B" is $c = -h_c/2 = _____\text{mm}$ (note that $c$ is positive outward from the center of radius).

The assumed normal bending stress at "B" for a straight beam assumption is $\sigma_{B,\text{straight}} = \frac{-M_B c}{I_{BC}} = _____\text{MPa}$.

(Confirm that the normal stress at "B" should be tension (i.e. $+\sigma$))

c) At "C", the moment is the same as at "B" such that $M_C= M_B = _____\text{N} \cdot \text{mm}$. (see 6b)

The moment of inertia at B-C is $I_{BC} = \frac{bh^3}{12} = _____\text{mm}^4$. (see 6b)

The distance from the assumed neutral axis to point "C" is $c = h_c/2 = _____\text{mm}$ (note that $c$ is positive outward from the center of radius).

The assumed normal bending stress at "C" for a straight beam assumption is $\sigma_{C,\text{straight}} = \frac{-M_C c}{I_{BC}} = _____\text{MPa}$.

(Confirm that the normal stress at "C" should be compression (i.e. $-\sigma$))
7) Analytically-Determined Stresses Using Curved Beam Relations

For the curved part of the beam (in this case points "B" and "C") the analytical calculation must take into account the initial curvature of the beam (Eq. 2).

Fill in the table with values for the loaded test specimen that are used in the following calculations.

<table>
<thead>
<tr>
<th>Applied Moment, ( M_B=M_C ) (N•mm)</th>
<th>Outer Radius, ( r_o ) (mm)</th>
<th>Inner Radius, ( r_i ) (mm)</th>
<th>Average radius, ( \bar{r} = (r_o + r_i)/2 ) (mm)</th>
<th>&quot;Height&quot; of Curve, ( h_C ) (mm)</th>
</tr>
</thead>
</table>

a) At the line in the curve connecting "B" and "C", the radius of the neutral axis for a rectangular cross section can be calculated from the outer radius, \( r_o \), and the inner radius \( r_i \), such that

\[
R = \left( \frac{r_o - r_i}{\ln\left( \frac{r_o}{r_i} \right)} \right) = \text{________mm.}
\]

The eccentricity, \( e \), can be calculated from the average radius of the centroid, \( \bar{r} \), and the radius of the neutral axis, \( R = \text{______mm} \) such that \( e = \bar{r} - R = \text{______mm} \).

The cross sectional area is calculated from the thickness, \( b \), and the curved beam height, \( h_C \), such that \( A = b \cdot h_C = \text{______mm}^2 \). (Note that the distance from the neutral axis to the point of interest is \( y = R - r \)).

b) At "B", \( r=r_i \), therefore \( y_B = R - r_i = \text{______mm} \).

The curved beam, normal bending stress at "B" is

\[
\sigma_{B \text{ curved}} = \frac{M_B y_B}{A\varepsilon(R - y_B)} = \text{______MPa.}
\]

c) At "C", \( r=r_o \), therefore \( y_C = R - r_o = \text{______mm} \).

The curved beam, normal bending stress at "C" is

\[
\sigma_{C \text{ curved}} = \frac{M_C y_C}{A\varepsilon(R - y_C)} = \text{______MPa.}
\]

8) Additional Axial Normal Stress Component

Because the bending moment at "B-C" is produced by a transverse force (that is, not a pure bending moment), the total normal stress at "B-C" has two components: a tensile axial (in the loading direction) stress and a tensile/compressive bending stress.

a) The tensile axial stress is calculated from the applied test force, \( F_t = \text{______N} \) and the cross sectional area, \( A = b \cdot h_C = \text{______mm}^2 \).

The axial tensile stress is

\[
\sigma_{\text{axial}} = \frac{F_t}{A} = \text{______MPa (Confirm that this axial normal stress is tension (i.e. } +\sigma)}
\]
9) Comparisons of Total Normal Stress (bending and axial) at "B" and "C"

Note that the stress values \( \sigma_A \), \( \sigma_B \) and \( \sigma_C \) are the stresses determined experimentally as calculated in Section 5.

a) At "B", the total calculated stress using the straight beam assumption is
\[
\sigma_B^{\text{total (straight)}} = \sigma_B^{\text{axial}} + \sigma_B^{\text{straight}} = \text{_______ MPa.}
\]
Percent difference between the actual photoelastically-measured stress and the calculated stress is 
\[
100 \frac{\sigma_B^{\text{total (straight)}} - \sigma_B}{\sigma_B} = \text{_______ \%}.
\]

b) At "B", the total calculated stress using the curved beam relation is
\[
\sigma_B^{\text{total (curved)}} = \sigma_B^{\text{axial}} + \sigma_B^{\text{curved}} = \text{_______ MPa.}
\]
Percent difference between the actual photoelastically-measured stress and the calculated stress is 
\[
100 \frac{\sigma_B^{\text{total (curved)}} - \sigma_B}{\sigma_B} = \text{_______ \%}.
\]

c) At "B", the numerically-determined (from the finite element analysis (FEA)) normal stress in the y-direction is \( \sigma_B^{\text{FEA}} = \text{_______ MPa.} \)
Percent difference between the actual photoelastically-measured stress and the numerically-determined stress is 
\[
100 \frac{\sigma_B^{\text{FEA}} - \sigma_B}{\sigma_B} = \text{_______ \%}.
\]

d) At "C", the total calculated stress using the straight beam assumption is
\[
\sigma_C^{\text{total (straight)}} = \sigma_C^{\text{axial}} + \sigma_C^{\text{straight}} = \text{_______ MPa.}
\]
Percent difference between the actual photoelastically-measured stress and the calculated stress is 
\[
100 \frac{\sigma_C^{\text{total (straight)}} - \sigma_C}{\sigma_C} = \text{_______ \%}.
\]

e) At "C", the total calculated stress using the curved beam relation is
\[
\sigma_C^{\text{total (curved)}} = \sigma_C^{\text{axial}} + \sigma_C^{\text{curved}} = \text{_______ MPa.}
\]
Percent difference between the actual photoelastically-measured stress and the calculated stress is 
\[
100 \frac{\sigma_C^{\text{total (curved)}} - \sigma_C}{\sigma_C} = \text{_______ \%}.
\]

f) At "C", the numerically-determined (from the finite element analysis (FEA)) normal stress in the y-direction is \( \sigma_C^{\text{FEA}} = \text{_______ MPa.} \)
Percent difference between the actual photoelastically-measured stress and the numerically-determined stress is 
\[
100 \frac{\sigma_C^{\text{FEA}} - \sigma_C}{\sigma_C} = \text{_______ \%}.
\]

10) Comparisons of Normal Stress (bending) at "A"

a) At "A", the total calculated stress using the straight beam relation is
\[
\sigma_A^{\text{total (straight)}} = \sigma_A^{\text{straight}} + 0 = \text{_______ MPa.}
\]
Percent difference between the actual photoelastically-measured stress and the calculated stress is 
\[
100 \frac{\sigma_A^{\text{total (straight)}} - \sigma_A}{\sigma_A} = \text{_______ \%}.
\]

b) At "A", the numerically-determined (from the finite element analysis (FEA)) is \( \sigma_A^{\text{FEA}} = \text{_______ MPa.} \)
Percent difference between the actual photoelastically-measured stress and the numerically-determined stress is 
\[
100 \frac{\sigma_A^{\text{FEA}} - \sigma_A}{\sigma_A} = \text{_______ \%}.
\]
11) Discussion of Analytical, Experimental, and Numerical Results

Comment on similarities and differences between the experimental (photoelasticity), the analytical (straight and curved beam relations) and the numerical (FEA) at points "A", "B", and "C." Does the straight-beam assumption give a conservative (i.e., over predict stresses) or non conservative (i.e., under predict stresses) result?
**Extra effort:** Compare the neutral axis positions in the curved part of the beam for the photoelastic and the FEA results. Are they quantitatively and qualitatively similar?

**Extra effort:** Using the fringe values from the photoelastic analysis, plot the stresses across the curved cross section from "B" to "C". Plot the results from the FEA analysis on the same plot. Finally, calculate the stresses from "B" (r=ri) to "C" (r=ro) for the relation

\[ \sigma = \frac{F}{A} + \frac{M_{cy}}{Ae(R-y)} = \frac{F}{A} + \frac{M_c(R-\tilde{r})}{Ae(R-(R-\tilde{r}))} \].

Compare the results. Is the stress vs. distance relation linear or non-linear? Would you expect the straight beam assumption to give a linear or non-linear relation? Would the straight beam assumption over or under predict stresses.